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Direct summands of serial modules

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Abstract

A module is serial if it is a direct sum of uniserial modules. In this paper we consider the following problem: Is every direct summand of a serial module serial? Positive results in several special cases are obtained. In particular, we show that every direct summand of a finite direct sum of copies of a uniserial module U is again a direct sum of copies of U. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the basic problems concerning direct sum decompositions of modules is the following: Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of modules each satisfying a certain property (P), is then any direct summand of M also a direct sum of modules satisfying (P)? Perhaps the most well-known theorem of this type is Kaplansky's Theorem [10], which asserts that if M is a direct sum of countably generated modules then any direct summand of M is also a direct sum of countably generated modules. Similar problems concerning direct summands of direct sums of indecomposable modules occur in a natural way, and many of them still remain open. In particular, it appears to be unknown whether the direct summands of a direct sum of modules with local endomorphism rings are direct sums of modules with local endomorphism rings.

Recall that a module M is called *uniserial* if its submodules are linearly ordered by inclusion, and is *serial* if it is a direct sum of uniserial modules. While there is a well-developed theory of serial rings, i.e., rings R for which both R_R and ${}_RR$ are serial, relatively little is known about the behaviour of direct sum decompositions of serial modules, in general. The uniqueness problem for decompositions of a module

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into a finite number of uniserial summands, posed by Warfield [15], has been settled recently in a satisfactory way in [5]: Krull–Schmidt fails, but a weak Krull–Schmidt theorem still holds for a finite direct sum of uniserial modules (see [4] for the infinite direct sum case). The crucial fact observed in [5], that the endomorphism ring of an arbitrary uniserial module has at most two maximal right ideals, suggests that there could be some similarities, in a sense, between direct sum decompositions of serial modules and direct sum decompositions of modules with local endomorphism rings.

In this paper we consider the following problem: If M is a serial module over an arbitrary ring R, are direct summands of M also serial? Equivalently, if M is a serial module over an arbitrary ring R, does any direct sum decomposition of M refine to a decomposition into uniserial direct summands? Though the problem is natural enough, it does not seem to have been treated in the literature so far. In this paper we present positive results in several special cases. In particular, we show that any direct summand of a finite direct sum of copies of a uniserial module U is also a direct sum of uniserial modules each isomorphic to U.

Throughout this paper we consider associative rings R with identity, and all modules are unitary right modules. For each R-module M_R the Jacobson radical of M_R will be denoted by $\mathrm{Rad}(M_R)$, and the Jacobson radical of the ring R will be denoted by $J(R)$. For a module M and an index set I, $M^{(I)}$ is the direct sum of $|I|$ copies of M.

2. Arbitrary direct sums of uniserials

A module U is *small* if for any direct sum $M=\bigoplus_{\lambda\in\Lambda} M_\lambda$ with projections π_λ and a homomorphism $f: U \to M$ we have $\pi_\lambda \circ f = 0$ for all but a finite number of indices λ (see [8] or [13]). The module U is called *σ-small* if it is a countable ascending union of small submodules [13].

Lemma 2.1. *Every uniserial module over any ring is σ-small.*

Proof. By Fuchs–Salce [8, Lemma 24] (cf. [4, Lemma 4.2]), every uncountably generated uniserial module is small. The result follows immediately. □

As observed by Warfield [13, p. 273], Kaplansky's proof of [10, Theorem 1] shows that any direct summand of a direct sum of σ-small modules is also a direct sum of σ-small modules. This fact and Lemma 2.1 imply that any direct summand of a serial module is a direct sum of σ-small modules. Therefore, our problem can be reduced to the problem of studying direct summands of a direct sum of countably many uniserial modules.

Following Crawley and Jónsson [3], a module M is said to have the *exchange property* if whenever M is a direct summand of a direct sum $A=\bigoplus_{i\in I} A_i$, there are submodules B_i of A_i such that $A = M \oplus (\bigoplus_{i\in I} B_i)$. It is well known that any quasi-injective module has the exchange property [7], in particular, any semisimple module has the exchange property.

Proposition 2.2. *Let $M = \bigoplus_{i \in I} U_i$ be a direct sum of uniserial modules with local endomorphism rings. Then any direct summand of M is also a direct sum of uniserial modules, each isomorphic to one of the U_i.*

Proof. By Lemma 2.1 the modules U_i are σ-small. Since they have local endomorphism rings, they have the exchange property [14, Proposition 1]. By Warfield [13, Theorem 7], if A is a direct sum of σ-small submodules having the exchange property, then any two direct sum decompositions of A have isomorphic refinements. In particular, every direct summand of $M = \bigoplus_{i \in I} U_i$ is a direct sum of uniserial modules each isomorphic to some U_i. □

Note that if the base ring R is either commutative or right noetherian, then every uniserial right R-module has a local endomorphism ring [5, Example 2.3], so that every direct summand of a serial right R-module is serial by Proposition 2.2.

Let U_R be a module with local endomorphism ring, and suppose that U is a direct summand of $M = P \oplus Q$. Let $\pi_P : M \to P$, $\pi_Q : M \to Q$ and $\pi_U : M \to U$ denote the canonical projections relative to the decomposition $M = P \oplus Q$ and a decomposition $M = U \oplus C$. Then $1_U = \pi_U|_P \pi_P|_U + \pi_U|_Q \pi_Q|_U$. Since $\mathrm{End}(U_R)$ is local, either $\pi_U|_P \pi_P|_U$ or $\pi_U|_Q \pi_Q|_U$ is an automorphism of U_R. Therefore, U is isomorphic to a direct summand of either P or Q. Now, we prove the key lemma of this paper, which shows a similar (but somewhat weaker) result for indecomposable modules whose endomorphism rings have at most two maximal right ideals.

Lemma 2.3. *Let R be an arbitrary ring and U an indecomposable right R-module whose endomorphism ring has at most two maximal right ideals. Suppose that $M = U \oplus C = P \oplus P$ for arbitrary modules M, C and P. Then U is isomorphic to a direct summand of P.*

Proof. Set $S = \mathrm{End}(U_R)$. If S is local, the result is proved in the remark above. Suppose that S has exactly two maximal right ideals K_1 and K_2. First we show that K_1 and K_2 are two-sided ideals of S. Note that S is a semilocal ring, so every right unit of S is a unit of S (see e.g. [11, Proposition 20.8]). Then $K_1 \cup K_2$ is precisely the set of non-units of S. Therefore, if K is any proper left ideal of S, we have $K \subseteq K_1 \cup K_2$. If there are elements $x \in K \backslash K_1$ and $y \in K \backslash K_2$, then $x + y \notin K_1$ and $x + y \notin K_2$, so $x + y \notin K_1 \cup K_2$, which is a contradiction because $x + y \in K$. Hence, either $K \subseteq K_1$ or $K \subseteq K_2$. Now for any element $x \in K_1$, consider the left ideal Sx. If $x \in K_1 \cap K_2 = J(S)$, obviously $Sx \subseteq J(S) \subseteq K_1$. If $x \in K_1 \backslash K_2$, then by the observation above either $Sx \subseteq K_1$ or $Sx \subseteq K_2$. The latter cannot happen because $x \notin K_2$. Therefore, in any case, $Sx \subseteq K_1$, i.e. K_1 is a two-sided ideal. Similarly, K_2 is a two-sided ideal. Since $J(S) = K_1 \cap K_2$, there is an injective canonical ring homomorphism $S/J(S) \to S/K_1 \times S/K_2$. This ring homomorphism is onto by the Chinese Remainder Theorem because $K_1 + K_2 = E$. Thus, $S/J(S)$ is the direct product of two division rings.

Set $E = \mathrm{End}(M_R)$. There is an equivalence $\mathrm{add}(M_R) \to \text{proj-}E$ of the category $\mathrm{add}(M_R)$ of all the R-modules isomorphic to direct summands of finite direct sums of copies of M_R into the category proj-E of all finitely generated projective right E-modules (the equivalence is given by $N_R \mapsto \mathrm{Hom}(M_R, N_R)$ for every $N \in \mathrm{add}(M_R)$). Note that M, P, U, C are objects of $\mathrm{add}(M_R)$. If we apply this equivalence to the direct sum decomposition $M = P \oplus P = U \oplus C$, we see that $\mathrm{Hom}(M_R, U)$ is isomorphic to a direct summand of E_E. More precisely, let $e \in E$ be the idempotent endomorphism of M_R such that $eM_R = U$. Then $\mathrm{Hom}(M_R, U) \cong eE$. It follows that (see e.g. [16, 22.2])

$$\begin{aligned} S/J(S) &= \mathrm{End}(U_R)/J(\mathrm{End}(U_R)) \cong \mathrm{End}(eE)/J(\mathrm{End}(eE)) \\ &\cong eEe/J(eEe) \cong \mathrm{End}(eE/eJ(E)). \end{aligned}$$

Note that eE is projective, so $eE/eJ(E)$ is a quasi-projective module (see e.g. [16, 18.2]). Since the endomorphism ring of $eE/eJ(E)$ is the direct product of two division rings, $eE/eJ(E) = L_1 \oplus L_2$ is the direct sum of two modules L_1 and L_2 whose endomorphism rings are division rings and $\mathrm{Hom}(L_1, L_2) = \mathrm{Hom}(L_2, L_1) = 0$. But the modules L_i $(i = 1, 2)$ are finitely generated and quasi-projective, so that they are local modules, i.e., they contain a unique maximal submodule (cf. [9, Corollary 4(2)]). Now, $\mathrm{Rad}(L_1) = \mathrm{Rad}(L_2) = 0$, hence, L_1 and L_2 are simple modules. Moreover, $\mathrm{Hom}(L_1, L_2) = 0$ implies that L_1 and L_2 are not isomorphic.

Since $M = P \oplus P$, the module $E_E = \mathrm{End}(M_R)$ is the direct sum of two right ideals D_1, D_2 of E both isomorphic to $\mathrm{Hom}(M_R, P_R)$. Set $\bar{D}_i = D_i/D_iJ(E)$ for $i = 1, 2$, so that $\bar{E} = E/J(E) = \bar{D}_1 \oplus \bar{D}_2$ and $\bar{D}_1 \cong \bar{D}_2$. Since $\bar{E} \cong eE/eJ(E) \oplus (1-e)E/(1-e)J(E)$, $L_1 \oplus L_2$ is a direct summand of $\bar{D}_1 \oplus \bar{D}_2$. By the exchange property of $L_1 \oplus L_2$, there are decompositions $\bar{D}_1 = \bar{D}_1' \oplus \bar{D}_1''$ and $\bar{D}_2 = \bar{D}_2' \oplus \bar{D}_2''$ such that $\bar{E} = L_1 \oplus L_2 \oplus \bar{D}_1' \oplus \bar{D}_2'$. It follows that $L_1 \oplus L_2 \cong \bar{D}_1'' \oplus \bar{D}_2''$. If $\bar{D}_1'' = 0$, we have $L_1 \oplus L_2 \cong \bar{D}_2''$. If $\bar{D}_2'' = 0$, we have $L_1 \oplus L_2 \cong \bar{D}_1''$. In both cases, $L_1 \oplus L_2$ is isomorphic to a direct summand of $\bar{D}_1 \cong \bar{D}_2$. Suppose that $\bar{D}_1''$ and $\bar{D}_2''$ are both non-zero. Since L_1 and L_2 are simple, by symmetry we can suppose that $L_1 \cong \bar{D}_1''$ and $L_2 \cong \bar{D}_2''$. But $\bar{D}_1 \cong \bar{D}_2$, hence L_2 is isomorphic to a direct summand T_2 of $\bar{D}_1 = \bar{D}_1' \oplus \bar{D}_1''$. Since T_2 is simple, it has the exchange property, so that there is a direct decomposition $\bar{D}_1' = \bar{C}_1' \oplus \bar{C}_1''$ such that either $\bar{D}_1 = T_2 \oplus \bar{C}_1'$ or $\bar{D}_1 = T_2 \oplus \bar{C}_1' \oplus \bar{D}_1''$. If the first equality holds, we have $T_2 \cong \bar{C}_1'' \oplus \bar{D}_1''$, and since T_2 and $\bar{D}_1''$ are simple and not isomorphic, we obtain a contradiction. If the second equality holds, it follows that $T_2 \cong \bar{C}_1''$, so that $L_1 \oplus L_2$ is isomorphic to the direct summand $\bar{D}_1'' \oplus \bar{C}_1'$ of $\bar{D}_1$.

Hence, in any case, $eE/eJ(E)$ is isomorphic to a direct summand of $D_1/D_1J(E)$. In particular, $eE/eJ(E)$ is a homomorphic image of D_1. Since eE is a projective cover of $eE/eJ(E)$ and D_1 is projective, by the uniqueness of projective covers (see e.g. [1, Lemma 17.17]), eE is isomorphic to a direct summand of D_1. In view of the equivalence above, this proves that U is isomorphic to a direct summand of P. □

As a consequence, we obtain the following proposition, which is crucial for proving most of the results in this paper.

Proposition 2.4. *Let U be a uniserial module over any ring R. Suppose that U is isomorphic to a direct summand of $P \oplus P$ for some R-module P. Then U is isomorphic to a direct summand of P.*

Proof. By [5, Theorem 1.2] the endomorphism ring of any uniserial module U either is local or has exactly two maximal right ideals I and K, where $I = \{ f \in \mathrm{End}(U) \mid f$ is not monic$\}$ and $K = \{ g \in \mathrm{End}(U) \mid g$ is not epic$\}$. Now the result follows from Lemma 2.3. □

Recall that a module M is *uniform* in case any two non-zero submodules of M have non-zero intersection. The next lemma is well known, but we include a short proof for the reader's convenience. In particular, from the lemma it will follow that every uniform submodule of a serial module is uniserial.

Lemma 2.5. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform modules and $\pi_i : M \to M_i$ be the canonical projections. If C is a uniform submodule of M, then there is an index $k \in I$ such that the restriction $\pi_k|_C : C \to M_k$ is a monomorphism.*

Proof. Consider the family T of all the subsets $J \subseteq I$ with the property that $(\bigoplus_{i \in J} M_i) \cap C = 0$. The family T is non-empty and partially ordered by inclusion. By Zorn's Lemma, there is a maximal element $J_0 \in T$. If $(C \oplus (\bigoplus_{i \in J_0} M_i)) \cap M_k = 0$ for some $k \in I$, this would imply that $C \cap ((\bigoplus_{i \in J_0} M_i) \oplus M_k) = 0$, a contradiction to the maximality of J_0. Hence, $(C \oplus (\bigoplus_{i \in J_0} M_i)) \cap M_k \neq 0$ for all $k \in I$. Since M_k is uniform for all $k \in I$, the module $C \oplus (\bigoplus_{i \in J_0} M_i)$ is essential in M. From this we deduce easily that $J_0 = I \setminus \{k\}$ for some index $k \in I$. Then the restriction $\pi_k|_C : C \to M_k$ of the natural projection π_k must be a monomorphism. □

We are now interested in the following question, which is a weakened form of the initial problem: If $M = \bigoplus_{i \in I} U_i$ is a direct sum of uniserial modules, does every non-zero direct summand of M contain a non-zero uniserial direct summand? Our next result sheds some light on this question in the special case in which all the U_i are isomorphic to each other.

Proposition 2.6. *Let U be a uniserial module and I be an arbitrary non-empty index set. Suppose that $U^{(I)} = A \oplus B$. Then either A or B must contain a direct summand isomorphic to U.*

Proof. The statement is trivial if A or B are zero, so that we may assume $A \neq 0$ and $B \neq 0$. Write $M = \bigoplus_{i \in I} U_i$, where $U_i \cong U$ for all $i \in I$. Let $\pi_i : M \to U_i$, $\pi_A : M \to A$ and $\pi_B : M \to B$ denote the natural projections corresponding to the decompositions $M = \bigoplus_{i \in I} U_i$ and $M = A \oplus B$. Fix an index $i \in I$. We have $1_{U_i} = \pi_i|_A \pi_A|_{U_i} + \pi_i|_B \pi_B|_{U_i}$, so that either $\pi_i|_A\pi_A|_{U_i}$ or $\pi_i|_B\pi_B|_{U_i}$ is an epimorphism by [5, Lemma 1.4(b)]. By symmetry we may suppose that $\pi_i|_A\pi_A|_{U_i}$ is an epimorphism. If $\pi_i|_A\pi_A|_{U_i}$ is an automorphism of U_i, then A contains a direct summand isomorphic to $U_i \cong U$, and we are done. Hence, we may suppose that $\pi_i|_A\pi_A|_{U_i}$ is not a monomorphism.

We claim that there is an index $j \in I$ such that the restriction $\pi_A|_{U_j}: U_j \to A$ is a monomorphism. This is equivalent to claiming that there is an index $j \in I$ such that $U_j \cap B = 0$. Suppose that such a j does not exist, i.e. $U_j \cap B \neq 0$ for all $j \in I$. Then it would follow that B is essential in M, hence $B = M$ and $A = 0$, a contradiction. This proves the claim. Note that $C = \pi_A(U_j)$ is uniform (in fact uniserial), hence by Lemma 2.5 there is an index $k \in I$ such that the restriction $\pi_k|_C : C \to U_k$ is a monomorphism. Since $U_i \cong U_j \cong U_k \cong U$, there are two homomorphisms $f : U_i \to A$ and $g : A \to U_i$ such that the composite map gf is a monomorphism. If gf is an isomorphism, A contains a direct summand isomorphic to U, and we are done. Therefore, we may assume that gf is not an epimorphism.

Since U_i is a uniserial module, the mapping $\pi_i|_A\pi_A|_{U_i} + gf$ is an automorphism of U_i [5, Lemma 1.4(a)]. Hence, the composite mapping of

$$(\pi_A|_{U_i}\ f): U_i \to A \oplus A \quad \text{and} \quad \begin{pmatrix} \pi_i|_A \\ g \end{pmatrix} : A \oplus A \to U_i$$

is an automorphism of U_i, so that U_i is isomorphic to a direct summand of $A \oplus A$. By Corollary 2.4, this implies that A contains a direct summand isomorphic to U_i. □

When the index set I is finite, Proposition 2.6 yields the answer to our question.

Theorem 2.7. *Let U be a uniserial module over an arbitrary ring and n be a natural number. Then any direct summand of U^n is isomorphic to U^m for some $m \leq n$.*

Proof. The proof proceeds by induction on n. If $n = 1$, the result holds trivially. Suppose that the result holds for all $k < n$, and let $U^n = A \oplus B$, where A and B are non-zero direct summands of U^n. By Proposition 2.6, either A or B must contain a direct summand isomorphic to U, say $A \cong U \oplus A'$. By the cancellation property of uniserial modules [5, Corollary 1.3], we get $U^{n-1} \cong A' \oplus B$. By the inductive hypothesis, $A' \cong U^r$ and $B \cong U^m$ for suitable $r, m \leq n-1$. Then $A \cong U^{r+1}$. □

Corollary 2.8. *Let R be an arbitrary ring, U_R a uniserial module and $E = \mathrm{End}(U_R)$ its endomorphism ring. Then every finitely generated projective right E-module is free.*

Proof. The corollary follows immediately from Theorem 2.7 and the equivalence between the category add(M_R) of all the R-modules isomorphic to direct summands of finite direct sums of copies of M_R and the category proj-E of all finitely generated projective right E-modules. □

In view of Corollary 2.8, it would be interesting to know whether all projective (right) modules over the endomorphism ring of a uniserial module U_R must be free.

Also, we do not know whether in Proposition 2.6, both A and B must contain a direct summand isomorphic to U.

Following Bass [2], a right R-module M is called *uniformly* $\aleph$*-big*, where $\aleph$ is an infinite cardinal number, provided M can be generated by $\aleph$ elements and M/MI requires $\aleph$ generators for all proper two-sided ideals I of R.

Proposition 2.9. *Let U be a uniserial right R-module and A a direct summand of a direct sum of copies of U. Then $A = A_1 \oplus A_2$, where A_1 is a direct sum of copies of U and A_2 has no maximal submodules.*

Proof. Suppose that U is not cyclic, and $U^{(I)} = A \oplus B$, where I is an index set. Since U is uniserial and not cyclic, we have $U = \mathrm{Rad}(U)$, hence $U^{(I)} = \mathrm{Rad}(U^{(I)}) = \mathrm{Rad}(A) \oplus \mathrm{Rad}(B)$ (see e.g. [1, Proposition 9.19]). It follows that $A \oplus B = \mathrm{Rad}(A) \oplus \mathrm{Rad}(B)$, hence $A = \mathrm{Rad}(A)$, i.e. A does not contain maximal submodules.

Hence, from now on, we may assume that U is a cyclic uniserial module. Set $E = \mathrm{End}(U_R)$. There is an equivalence $H : \mathrm{Add}(U_R) \to \mathrm{Proj}\text{-}E$ of the category $\mathrm{Add}(U_R)$ of all the R-modules isomorphic to direct summands of direct sums of copies of U_R into the category Proj-E of all projective right E-modules (the equivalence is given by $N_R \mapsto \mathrm{Hom}(U_R, N_R)$ for every $N \in \mathrm{Add}(U_R)$; see [1, Lemma 29.4]). Since $H(A) \in \mathrm{Proj}\text{-}E$, $H(A)$ is a direct sum of countably generated projective right E-modules by Kaplansky's theorem [10]. Therefore, without loss of generality, we may suppose that A is a direct summand of $U^{(\aleph_0)}$. If A is finitely generated, then A is a direct summand of U^n for some integer $n \geq 1$, so $A \cong U^m$ for some $m \geq 1$ by Theorem 2.7. Therefore, we assume that $P = H(A)$ is a countably generated projective right E-module that is not finitely generated. By [5, Theorem 1.2] the ring E has at most two maximal right ideals. If E is a local ring, the result follows by Proposition 2.2. Hence, we may assume that E has exactly two maximal right ideals I and K, where $I = \{ f \in \mathrm{End}(U) \mid f \text{ is not monic}\}$ and $K = \{ g \in \mathrm{End}(U) \mid g \text{ is not epic}\}$. Denote by J the Jacobson radical of E, and consider the vector space P/PI over E/I and the vector space P/PK over E/K. If both of them are infinite-dimensional vector spaces, then for every two-sided ideal L of R, L is contained either in I or in K, so that P/PL is not finitely generated. Hence, P is uniformly $\aleph_0$-big as a right E-module, which implies by Bass [2, Theorem 3.1] (cf. also Rowen [12, Theorem 5.1.67]) that P is a free E-module. In this case A is a direct sum of copies of U, and we are done.

Therefore, it suffices to consider the case in which one of the two dimensions $n_1 = \dim_{E/I}(P/PI)$ and $n_2 = \dim_{E/K}(P/PK)$ is finite. Set $n = \min\{n_1, n_2\}$. On the one hand, we have

$$P/PJ \cong P/PI \oplus P/PK \cong (E/I)^{n_1} \oplus (E/K)^{n_2}.$$

On the other hand, $E^n/E^nJ \cong (E/I)^n \oplus (E/K)^n$, so there is an epimorphism $f : P/PJ \to E^n/E^nJ$, which implies that E^n/E^nJ is an epimorphic image of P. But n is finite, so E^n is a projective cover of E^n/E^nJ. Hence, by the uniqueness of projective covers (e.g. [1, Lemma 17.17]), E^n is isomorphic to a direct summand of P, so $P \cong E^n \oplus Q$

for some direct summand Q of P. Thus, A has a decomposition $A = A_1 \oplus A_2$, where $A_1 \cong U^n$ and $H(A_2) = Q$. Since $n = \min\{n_1, n_2\}$, it follows that either $\dim_{E/I} Q/QI = 0$ or $\dim_{E/K} Q/QK = 0$, i.e. either $Q = QI$ or $Q = QK$.

Suppose that $Q = QK$. Since $Q = \mathrm{Hom}(U_R, A_2)$, for every homomorphism $f : U \to A_2$ there is a finite number of homomorphisms $f_1, \dots, f_n : U \to A_2$ and $g_1, \dots, g_n \in K$ such that $f = \sum_{i=1}^n f_i g_i$. Every $g_i : U \to U$ is not surjective, so that $g_i(U) \subseteq \mathrm{Rad}(U)$. Then $f_i g_i(U) \subseteq f_i(\mathrm{Rad}(U)) \subseteq \mathrm{Rad}(A_2)$, which implies that $f(U) \subseteq \mathrm{Rad}(A_2)$. But A_2 is a direct summand of $U^{(\aleph_0)}$, hence U generates A_2, so $A_2 = \sum_{f \in Hom(U, A_2)} f(U) \subseteq \mathrm{Rad}(A_2)$. Therefore, $A_2 = \mathrm{Rad}(A_2)$, so that A_2 has no maximal submodules.

Finally, suppose that $Q = QI$. Then for every homomorphism $f : U \to A_2$ there is a finite number of homomorphisms $f_1, \dots, f_n : U \to A_2$ and $g_1, \dots, g_n \in I$ such that $f = \sum_{i=1}^n f_i g_i$. Since every $g_i \in I$, that is, $g_i : U \to U$ is not injective, and U is uniserial, it follows that f is not injective. Let $\pi : U^{(\aleph_0)} \to A_2$ be the splitting canonical projection and $\varepsilon_i : U_i \to U^{(\aleph_0)}$ be the canonical injections. Then $\bigoplus_{i \in \aleph_0} \ker(\pi\varepsilon_i) \subseteq \ker(\pi)$, and each $\ker(\pi\varepsilon_i)$ is non-zero, which implies that $\ker(\pi)$ is essential in $U^{(\aleph_0)}$. But $\ker(\pi) \cap A_2 = 0$, therefore $A_2 = 0$, and so $A = A_1 \cong U^n$. □

Note that if $M = \bigoplus_{i \in I} M_i$ is a direct sum of modules M_i with local endomorphism rings, then by Azumaya's theorem (e.g. [1, Theorem 12.6]) every non-zero direct summand of M contains an indecomposable direct summand isomorphic to some M_i. In view of this fact, our next corollary may be considered as a partial generalization of Proposition 2.2 (in the case $U_i \cong U_j$ for all $i, j \in I$). It is interesting to remark that the two proofs are quite different.

Corollary 2.10. *Let U be a cyclic uniserial module and I an arbitrary index set. Suppose that every non-zero direct summand of $U^{(I)}$ contains a direct summand isomorphic to U. Then every direct summand of $U^{(I)}$ is a direct sum of copies of U.*

Proof. Let A be a direct summand of $U^{(I)}$. By Proposition 2.9 we have $A = A_1 \oplus A_2$, where A_1 is a direct sum of copies of U, and A_2 has no maximal submodules. Then every direct summand of A_2 has no maximal submodules. If $A_2 \neq 0$, by hypothesis A_2 contains a direct summand isomorphic to U. Since U is cyclic, U contains a maximal submodule. Thus, we get a contradiction, which shows that $A_2 = 0$. Hence, $A = A_1$ is a direct sum of copies of U. □

We conclude this section with a remark about the so-called "$\aleph$th root uniqueness" for uniserial modules. It was shown in [6] that modules with semilocal endomorphism rings satisfy the nth root uniqueness property, i.e. if M and N are modules with $\mathrm{End}(M)$ and $\mathrm{End}(N)$ semilocal and n is a positive integer, then $M^n \cong N^n$ implies $M \cong N$. It is natural to ask if a similar "$\aleph$th root uniqueness" holds for modules with semilocal endomorphism rings, where $\aleph$ is an arbitrary cardinal number. Lawrence Levy has recently communicated to us an example (unpublished) showing that there exist indecomposable modules M and N with $\mathrm{End}(M)$ and $\mathrm{End}(N)$ semilocal such that

$M^{(\aleph_0)} \cong N^{(\aleph_0)}$, but M is not isomorphic to N. However if one of M or N is cyclic uniserial, we do have "$\aleph$th root uniqueness" for an arbitrary cardinal number $\aleph \neq 0$.

Proposition 2.11. *Let U be a cyclic uniserial module and N an indecomposable module, and suppose that $U^{(\aleph)} \cong N^{(\aleph)}$ for some cardinal number $\aleph \neq 0$. Then $U \cong N$.*

Proof. Since U is cyclic and $U^{(\aleph)} \cong N^{(\aleph)}$, U is isomorphic to a direct summand of a finite direct sum N^n of copies of N. Let m be an integer with $n \leq 2^m$, so that U is isomorphic to a direct summand of N^{2^m}. Now apply Proposition 2.4 m times. It follows that $U \cong N$. □

3. Finite direct sums of uniserials

In this section we concentrate on finite direct sums of uniserial modules.

Proposition 3.1. *Let $U_1, \ldots, U_n$ be uniserial modules, and suppose that $M = U_1 \oplus \cdots \oplus U_n \cong P \oplus P$ for some module P. Then P is serial.*

Proof. Induction on $n \geq 0$. The cases $n \leq 2$ are trivial (for $n = 2$ recall that every uniform submodule of the serial module M is uniserial by Lemma 2.5).

Suppose that $M = U_1 \oplus \cdots \oplus U_n \cong P \oplus P$ with $n \geq 3$. By Proposition 2.4 the uniserial module U_1 is isomorphic to a direct summand of P, $P \cong U_1 \oplus P'$ say. Then $U_1 \oplus \cdots \oplus U_n \cong U_1 \oplus U_1 \oplus P' \oplus P'$, so that $U_2 \oplus \cdots \oplus U_n \cong U_1 \oplus P' \oplus P'$ by cancellation [5, Corollary 1.3]. By [5, Proposition 1.5] there are two distinct indices $i, j = 2, 3, \ldots, n$ such that U_1 is isomorphic to a direct summand of $U_i \oplus U_j$. For simplicity of notation suppose $i = 2$ and $j = 3$, so that $U_2 \oplus U_3 \cong U_1 \oplus W$ for a suitable uniserial module W. Then $U_1 \oplus W \oplus U_4 \oplus \cdots \oplus U_n \cong U_1 \oplus P' \oplus P'$ implies that $W \oplus U_4 \oplus \cdots \oplus U_n \cong P' \oplus P'$. By the inductive hypothesis P', hence P also, are serial modules. □

We observe a technical lemma.

Lemma 3.2. *Let M be a module with two decompositions $M = U_1 \oplus \cdots \oplus U_n = A \oplus B$, where $U_1, \ldots, U_n$ are uniserial modules and $A \neq 0$. Let $\pi_i : M \to U_i$, $i = 1, \ldots, n$, and $\pi_A : M \to A$ denote the canonical projections corresponding to these decompositions. Then there are two indices $k, t = 1, \ldots, n$ such that the composite map $\pi_t|_A \pi_A|_{U_k} : U_k \to A \to U_t$ is an epimorphism.*

Proof. Clearly $A \subseteq \bigoplus_{i=1}^n \pi_i(A)$. Suppose $\pi_i(A) \neq U_i$ for all $i = 1, \ldots, n$. Then $\pi_i(A)$ is superfluous in U_i, hence superfluous in M. It follows that $\bigoplus_{i=1}^n \pi(A)$ is superfluous in M, so A also is superfluous in M. But this is a contradiction because $M = A \oplus B$. This shows that there is an index $t = 1, \ldots, n$ such that $\pi_t(A) = U_t$, i.e. $U_t = \pi_t|_A(A) = \sum_{i=1}^n \pi_t|_A \pi_A(U_i)$. But U_t is uniserial, so there is an index $k = 1, \ldots, n$

such that $U_t = \pi_t|_A \pi_A(U_k)$. This shows that the composite map of $\pi_A|_{U_k} : U_k \to A$ and $\pi_t|_A : A \to U_t$ is an epimorphism. □

Following [5], two modules A and B are said to belong to *the same monogeny class*, written $[A]_m = [B]_m$, if there are a monomorphism $f : A \to B$ and a monomorphism $g : B \to A$. Similarly, A and B are in *the same epigeny class*, written $[A]_e = [B]_e$, if there are an epimorphism $h : A \to B$ and an epimorphism $\ell : B \to A$.

Theorem 3.3. *Let $U_1, \dots, U_n$ be uniserial modules. Suppose that for any pair U_i and U_j there exists a uniserial module W such that $[U_i]_m = [W]_m$ and $[U_j]_e = [W]_e$. Then any direct summand of $U_1 \oplus \cdots \oplus U_n$ is serial.*

Proof. Assume that $M = U_1 \oplus \cdots \oplus U_n = A \oplus B$ with A and B non-zero. Let $\pi_i : M \to U_i$, $\pi_A : M \to A$ and $\pi_B : M \to B$ denote the canonical projections corresponding to the decompositions $M = \bigoplus_{i=1}^n U_i$ and $M = A \oplus B$, respectively.

We claim that either A or B has a non-zero uniserial direct summand. In order to prove the claim note that $1_{U_1} = \pi_1|_A \pi_A|_{U_1} + \pi_1|_B \pi_B|_{U_1}$. Hence, either $\pi_1|_A \pi_A|_{U_1}$ or $\pi_1|_B \pi_B|_{U_1}$ is injective [5, Lemma 1.4(b)]. By symmetry we may suppose, without loss of generality, that $\pi_1|_A \pi_A|_{U_1}$ is injective. Now, we prove that there is an index k and homomorphisms $\alpha : U_k \to A$ and $\beta : A \to U_k$ such that the composite map $\beta\alpha$ is a surjective endomorphism of U_k. By Lemma 3.2 there are U_k and U_t such that the composite map $\pi_t|_A \pi_A|_{U_k} : U_k \to A \to U_t$ is an epimorphism. By hypothesis there is a uniserial module V such that $[U_t]_m = [V]_m$ and $[U_k]_e = [V]_e$. This implies that there are a monomorphism $f : V \to U_t$ and an epimorphism $g : V \to U_k$. Then $h = \pi_t|_A \pi_A|_{U_k} g : V \to U_t$ is an epimorphism. Since V and U_t are uniserial, it follows from [5, Lemma 1.4(a)] that either f or h or $f + h$ is an isomorphism. In all the three cases we get that $V \cong U_t$, hence $[U_k]_e = [U_t]_e$. Let $\ell : U_t \to U_k$ be an epimorphism. Then the maps $\alpha = \pi_A|_{U_k} : U_k \to A$ and $\beta = \ell\pi_t|_A : A \to U_k$ have the property that the composite map $\beta\alpha$ is a surjective endomorphism of U_k.

By hypothesis there is a uniserial module W such that $[U_1]_m = [W]_m$ and $[U_k]_e = [W]_e$. Since $\pi_1|_A \pi_A|_{U_1} : U_1 \to A \to U_1$ is injective and $\beta\alpha : U_k \to A \to U_k$ is surjective, there are maps $f_1, g_1 \in \mathrm{Hom}(W, A)$ and $f_2, g_2 \in \mathrm{Hom}(A, W)$ such that the composite map $f_2 f_1$ is injective and the composite map $g_2 g_1$ is surjective. If $f_2 f_1$ is an automorphism of W, then A has a direct summand isomorphic to W, and we are done. Similarly, if $g_2 g_1$ is an automorphism of W, again A has a direct summand isomorphic to W. If neither $f_2 f_1$ nor $g_2 g_1$ are automorphisms, then $f_2 f_1 + g_2 g_1$ is an automorphism of W [5, Lemma 1.4(a)], hence, W is isomorphic to a direct summand of $A \oplus A$. By Proposition 2.4 it follows that W is isomorphic to a direct summand of A. This proves our claim.

Now the proof of the theorem is by induction on n. The case $n = 0$ is trivial. If $M = U_1 \oplus \cdots \oplus U_n = A \oplus B$, either A or B has a non-zero uniserial direct summand by the claim. By symmetry, we may assume that $A = U \oplus A'$, where U is non-zero uniserial. If U is isomorphic to some U_i, say $U \cong U_1$, then by the cancellation property of uniserial modules [5, Corollary 1.3] we get $U_2 \oplus \cdots \oplus U_n \cong A' \oplus B$, which implies by the inductive hypothesis that A' and B, and hence A and B, are serial. Now, suppose that U is not isomorphic to any of the U_i, $i = 1, \ldots, n$. By [5, Proposition 1.5], there are distinct indices, say 1 and 2 for simplicity, such that $U_1 \oplus U_2 \cong U \oplus U'$ for some submodule U'. Clearly U' is not isomorphic to either U_1 or U_2, and so by [5, Proposition 1.7] we have, without loss of generality, that $[U']_{\mathrm{m}} = [U_1]_{\mathrm{m}}$ and $[U']_{\mathrm{e}} = [U_2]_{\mathrm{e}}$. Since

$$U_1 \oplus U_2 \oplus \cdots \oplus U_n \cong U \oplus U' \oplus U_3 \oplus \cdots \oplus U_n \cong U \oplus A' \oplus B,$$

by the cancellation property we obtain

$$U' \oplus U_3 \oplus \cdots \oplus U_n \cong A' \oplus B.$$

Using the fact that $[U']_{\mathrm{m}} = [U_1]_{\mathrm{m}}$ and $[U']_{\mathrm{e}} = [U_2]_{\mathrm{e}}$, it is easy to check that the serial module $U' \oplus U_3 \oplus \cdots \oplus U_n$ also satisfies the hypotheses of the theorem. By the inductive hypothesis, this implies that A' and B, and hence A and B, are serial, which completes the induction. □

From Theorem 3.3 we obtain the following corollary, which is a generalization of Theorem 2.7.

Corollary 3.4. *Let $U_1, U_2, \ldots, U_n$ be uniserial modules. Suppose that either*
(a) $[U_i]_{\mathrm{m}} = [U_j]_{\mathrm{m}}$ *for each pair U_i and U_j, or*
(b) $[U_i]_{\mathrm{e}} = [U_j]_{\mathrm{e}}$ *for each pair U_i and U_j.*
Then any direct summand of $U_1 \oplus \cdots \oplus U_n$ is a direct sum of uniserial modules each isomorphic to some U_i.

Proof. For each pair U_i and U_j, if $[U_i]_{\mathrm{m}} = [U_j]_{\mathrm{m}}$, then setting $W = U_j$, we get $[U_i]_{\mathrm{m}} = [W]_{\mathrm{m}}$ and $[U_j]_{\mathrm{e}} = [W]_{\mathrm{e}}$. Similarly, if $[U_i]_{\mathrm{e}} = [U_j]_{\mathrm{e}}$, then setting $W = U_i$, we get $[U_i]_{\mathrm{m}} = [W]_{\mathrm{m}}$ and $[U_j]_{\mathrm{e}} = [W]_{\mathrm{e}}$. From Theorem 3.3 it follows that any direct summand of $M = U_1 \oplus \cdots \oplus U_n$ is serial. Now, if U is any uniserial direct summand of M, by [5, Proposition 1.7] either U is isomorphic to some U_i, or there are indices $i \neq j$ such that $[U]_{\mathrm{m}} = [U_i]_{\mathrm{m}}$ and $[U]_{\mathrm{e}} = [U_j]_{\mathrm{e}}$. If the hypothesis (a) holds, we have $[U]_{\mathrm{m}} = [U_i]_{\mathrm{m}} = [U_j]_{\mathrm{m}}$, hence $U \cong U_j$ by [5, Proposition 1.6]. Similarly, if the hypothesis (b) holds, we get $U \cong U_i$. □

Let $U_1, \ldots, U_n$ be uniserial modules. Note that, on the one hand, if one of the hypotheses (a) and (b) of Corollary 3.4 is satisfied, then the Krull–Schmidt Theorem holds for the module $U_1 \oplus \cdots \oplus U_n$. On the other hand, if $U_i \oplus U_j$ does not satisfy the Krull–Schmidt Theorem for each pair i and j, that is, $U_i \oplus U_j = V_{ij} \oplus W_{ij}$ with V_{ij} and W_{ij} not isomorphic to U_i and U_j, then the hypotheses of Theorem 3.3 are satisfied (cf. [5, Proposition 1.7]). This motivates our next result which narrows down the class of possible counterexamples.

Define a uniserial module U to be *mono-Krull–Schmidt* if for every module V, $[U]_m=[V]_m$ implies $U\cong V$. Similarly, define a uniserial module U to be *epi-Krull–Schmidt* if for every module V, $[U]_e=[V]_e$ implies $U\cong V$. Thus, a uniserial module U is Krull–Schmidt [5, Definition 1.10] if and only if it is either mono-Krull–Schmidt or epi-Krull–Schmidt.

Proposition 3.5. *Let $U_1,U_2,\dots,U_n$ be mono-Krull–Schmidt uniserial modules. Suppose that $[U_i]_e\neq[U_j]_e$ for every $i,j=1,2,\dots,n$, $i\neq j$. Then every direct summand of $U_1\oplus U_2\oplus\dots\oplus U_n$ is serial.*

Proof. Without loss of generality, we may suppose $U_i\neq 0$ for every $i=1,2,\dots,n$.

The proof will be by induction on n. The case $n=1$ is trivial. Suppose $M=U_1\oplus U_2\oplus\cdots\oplus U_n=P\oplus Q$. Let $\pi_i:M\to U_i$, $\pi_P:M\to P$, $\pi_Q:M\to Q$, $\varepsilon_i:U_i\to M$, $\varepsilon_P:P\to M$, $\varepsilon_Q:Q\to M$ denote the canonical projections and embeddings corresponding to these direct sum decompositions of M.

Assume first that both P and Q have no non-zero uniserial direct summand. We claim that, under this hypothesis, for every $i=1,2,\dots,n$ there exists an epimorphism $U_j\to U_i$ for some index $j=1,2,\dots,n$, $j\neq i$.

In order to prove the claim, fix $i=1,2,\dots,n$. Say $i=1$ for simplicity of notation. Then $1_{U_1}=\pi_1\varepsilon_1=\pi_1 1_M\varepsilon_1=\pi_1(\varepsilon_P\pi_P+\varepsilon_Q\pi_Q)\varepsilon_1=\pi_1\varepsilon_P\pi_P\varepsilon_1+\pi_1\varepsilon_Q\pi_Q\varepsilon_1$. If one of these two summands is an automorphism of U_1, say $\pi_1\varepsilon_P\pi_P\varepsilon_1$ is an automorphism of U_1, then U_1 is isomorphic to a direct summand of P, a contradiction. Hence, neither $\pi_1\varepsilon_P\pi_P\varepsilon_1$ nor $\pi_1\varepsilon_Q\pi_Q\varepsilon_1$ is an automorphism of U_1. Then one of $\pi_1\varepsilon_P\pi_P\varepsilon_1$, $\pi_1\varepsilon_Q\pi_Q\varepsilon_1$ is injective and not surjective, and the other is surjective and not injective [5, Lemma 1.4(b)]. By symmetry, we may suppose, without loss of generality, that $\pi_1\varepsilon_P\pi_P\varepsilon_1$ is injective and not surjective, and $\pi_1\varepsilon_Q\pi_Q\varepsilon_1$ is surjective and not injective. Consider the idempotent endomorphism $\varepsilon_P\pi_P$ of $M=U_1\oplus\cdots\oplus U_n$. If we write it in matrix form,

$$\varepsilon_P\pi_P=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\dots&\alpha_{1n}\\ \alpha_{21}&\alpha_{22}&\dots&\alpha_{2n}\\ \vdots&\vdots&&\vdots\\ \alpha_{n1}&\alpha_{n2}&\dots&\alpha_{nn}\end{pmatrix},$$

where $\alpha_{ij}:U_j\to U_i$, then $\pi_1\varepsilon_P\pi_P\varepsilon_1=\alpha_{11}$. Since $\varepsilon_P\pi_P$ is idempotent, it follows that $\alpha_{11}=\sum_{i=1}^n\alpha_{1i}\alpha_{i1}$. Hence $\alpha_{11}(1-\alpha_{11})=\sum_{i=2}^n\alpha_{1i}\alpha_{i1}$. As $\alpha_{11}=\pi_1\varepsilon_P\pi_P\varepsilon_1$ is injective and not surjective, $1-\alpha_{11}$ is surjective. Hence $\operatorname{im}\alpha_{11}=\operatorname{im}(\alpha_{11}(1-\alpha_{11}))=\operatorname{im}(\sum_{i=2}^n\alpha_{1i}\alpha_{i1})\subseteq\sum_{i=2}^n(\operatorname{im}\alpha_{1i})$. Since U_1 is uniserial, it follows that there exists $j\neq 1$ such that $\operatorname{im}\alpha_{11}\subseteq\operatorname{im}\alpha_{1j}$. Therefore, the monomorphism $\alpha_{11}:U_1\to U_1$ induces a monomorphism $U_1\to\operatorname{im}\alpha_{1j}$ (it is sufficient to restrict the codomain to $\operatorname{im}\alpha_{1j}$); and the embedding $\operatorname{im}\alpha_{1j}\to U_1$ is obviously a monomorphism. Thus $[\operatorname{im}\alpha_{1j}]_m=[U_1]_m$. But U_1 is a mono-Krull–Schmidt uniserial module, so that $\operatorname{im}\alpha_{1j}\cong U_1$. Since $\alpha_{1j}:U_j\to U_1$ induces a surjective mapping $U_j\to\operatorname{im}\alpha_{1j}$, it follows that there exists a surjective mapping $U_j\to U_1$. This concludes the proof of the claim.

By the claim, if we start from $i_0 = 1$, we can construct an infinite sequence of epimorphisms

$$\cdots \to U_{i_2} \to U_{i_1} \to U_{i_0}$$

with $i_t \neq i_{t+1}$ for every $t \geq 0$. Since the set $\{1,2,\dots,n\}$ is finite, there exist two indices $t < u$ such that $U_{i_t} = U_{i_u}$. Hence, there is a sequence of epimorphisms

$$U_{i_u} \to U_{i_{u-1}} \to \cdots \to U_{i_{t+1}} \to U_{i_t} = U_{i_u}.$$

It follows that $[U_{i_u}]_{\mathrm{e}} = [U_{i_{u-1}}]_{\mathrm{e}} = \cdots = [U_{i_{t+1}}]_{\mathrm{e}} = [U_{i_t}]_{\mathrm{e}}$. In particular, $[U_{i_{t+1}}]_{\mathrm{e}} = [U_{i_t}]_{\mathrm{e}}$ and $i_{t+1} \neq i_t$. This is a contradiction. The contradiction shows that either P or Q must always have a non-zero uniserial direct summand.

Suppose, for example, that P has a non-zero uniserial direct summand, say $P = V \oplus P'$. Then V is a direct summand of $U_1 \oplus \cdots \oplus U_n$, so that $V \cong U_i$ for some i by [5, Proposition 1.7]. Since cancellation holds, $P' \oplus Q$ is isomorphic to the direct sum of all the modules U_i with $i \neq j$. By the inductive hypothesis both P' and Q are serial modules. Therefore, P is serial as well. □

Proposition 3.6. *Let $U_1, U_2, \dots, U_n$ be epi-Krull–Schmidt uniserial modules. Suppose that $[U_i]_{\mathrm{m}} \neq [U_j]_{\mathrm{m}}$ for every $i, j = 1,2,\dots,n$, $i \neq j$. Then every direct summand of $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ is serial.*

Proof. Dual to the previous one. □

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